



## Two dimensions are easier

B. KAWOHL

**Abstract.** In this little note I first recall a particularly short proof of the classical isoperimetric inequality in two dimensions. Other geometric inequalities are still open in more than two dimensions. I point out six of those.

**Mathematics Subject Classification.** Primary 49Q10; Secondary 51M16, 52A40, 31C45, 31B99.

**Keywords.** Geometric inequalities, Cheeger set, Meissner bodies, Ulam floating, Euler elastica, Farthest point distance function.

The classical isoperimetric inequality states that among sets of given perimeter, the ball contains maximal volume. Equivalently, a ball of given volume has minimal perimeter among competing sets of same volume. A simple proof of this in two dimensions, which breaks down in three or more dimensions, goes as follows. Without loss of generality, we may assume that the set of minimal perimeter is bounded, otherwise its perimeter is infinite, and connected, otherwise we translate its components until they touch each other. Moreover, in two dimensions the set of minimal perimeter is simply connected, because “filling any holes” reduces perimeter and increases area. Finally, in two dimensions we can take the convex hull of a minimizing domain, which does not increase perimeter nor decrease the enclosed area, and see that optimal domains must be convex. Both of these arguments fail in three dimensions for a torus. Once the optimal domain is known to be convex, its boundary can be locally represented as a Lipschitz function and the existence of an optimal domain follows from Blaschke’s selection theorem. From standard arguments in the calculus of variations, its boundary must have constant curvature, because in polar coordinates we maximize the area

---

This manuscript is dedicated to Ernst-Ulrich Gekeler with gratitude for his service to Archiv der Mathematik.

$$A(\Omega) = \int_0^{2\pi} \int_0^{r(\theta)} \rho d\rho d\theta = \frac{1}{2} \int_0^{2\pi} r^2(\theta) d\theta$$

subject to given length

$$L(\partial\Omega) := \int_0^{2\pi} \sqrt{r^2(\theta) + r_\theta^2(\theta)} d\theta.$$

So among periodic functions  $r(\theta)$ , we can look at the functional  $A + \lambda L$  with Euler-Lagrange equation

$$r + \frac{\lambda r}{\sqrt{r^2 + r_\theta^2}} - \frac{d}{d\theta} \left( \frac{\lambda r_\theta}{\sqrt{r^2 + r_\theta^2}} \right) = 0.$$

After integration

$$\frac{r r_{\theta\theta} - 2r_\theta^2 - r^2}{(r^2 + r_\theta^2)^{3/2}} = \frac{1}{\lambda},$$

where the left-hand side can be identified as curvature  $\kappa$ . Constant  $\kappa$  implies that the optimal domain is a disk.

**Open Problem 1:** The **Cheeger set**  $\Omega_C$  of an open bounded connected set  $\Omega$  is defined as a set that minimizes the ratio  $\frac{|\partial D|}{|D|}$  of perimeter  $|\partial D|$  over volume  $|D|$  among all subsets of  $\Omega$ . When  $\Omega$  is a square, the corresponding Cheeger set is a rounded square and can be calculated by elementary means [13]. Incidentally, the Cheeger constant  $\frac{|\partial\Omega_C|}{|\Omega_C|}$  given in [13] contains a typo. It should be  $(4-\pi)/\{2(a+b-\sqrt{(a-b)^2+\pi ab})\}$  for a rectangle  $(-a, a) \times (-b, b)$  with  $0 < a \leq b$  and  $(\sqrt{\pi}+2)/(2a)$  for a square  $(-a, a)^2$ . I thank Linda Wirth for pointing this out to me. When  $\Omega$  is a cube, no explicit analytical description of its Cheeger set has been given, other than that it is convex and that those parts of its boundary that do not touch the cube have constant mean curvature  $\frac{|\Omega_C|}{|\partial\Omega_C|}$ . A numerical approximation and visualization can be found in [11]

**Open Problem 2:** Convex sets of constant width [2, 3, 14] have been studied for more than a century. A nice exposition can be found in the book “Geometry and the Imagination” by Hilbert and Cohn-Vossen [10]. Among all two-dimensional convex sets of constant width  $d$ , the disk with radius  $d/2$  maximizes area and the Reuleaux-triangle minimizes area. A Reuleaux-triangle is the intersection of three disks with centers in the corners of an equilateral triangle. In three dimensions it has been shown that the ball maximizes volume among all convex sets of given width  $d$ , and it has been conjectured that the **Meissner-bodies** minimize volume. These are obtained from a small modification of the Meissner-tetrahedron, which is the intersection of four balls of radius  $d/2$  with centers in the four corners of a regular tetrahedron. For details I refer to [16].

**Open Problem 3:** Imagine a convex piece of land that you want to cut into two subsets of equal area with a minimal cut. Given the total area (but not

the shape) of the initial set, which shape renders **the longest shortest cut**? This problem was posed by Polyá in 1958 and his conjecture that the answer is a disk was not confirmed until 2012 in [4]. The proof is quite technical and the three-dimensional analogue that a ball and a bisecting plane will serve the same purpose seems to be a difficult open problem.

**Open Problem 4:** When a ball of specific weight  $1/2$  is dropped into water, in contrast to a cube it swims semistable in any direction. Is the ball the only shape that has this property, known as **Ulam floating**? Although the problem was widely circulated in the 1930's, there are still opposing convictions how to answer this question [19]. The problem is easier and was settled in two dimensions about 80 years ago. Cylindrical logs of constant two-dimensional cross-section  $D$  and specific weight  $1/2$  do not need to have a disk as their cross section to swim metastable in any orientation parallel to the axis of the cylinder. Any so-called Zindler set (and this includes certain heart-shaped sets  $D$ ) will serve the same purpose [1]. By definition, Zindler sets have the remarkable property that any line segment dividing the set into two subsets of equal area has the same length, independent of its direction. Among convex Zindler sets of given area, one can look for the one with the longest water-line dividing it into two sets of equal area. In view of Open Problem 3, one should suspect the disk as optimal, but this is wrong. In fact, the so-called Auerbach triangle is optimal [7]. This is surprising even to experts in shape optimization.

**Open Problem 5: Euler Elastica** are curves  $\gamma$  in the plane whose elastic energy is measured by  $\int_{\gamma} \kappa^2 ds$ , where  $\kappa$  denotes curvature. Similar to the isoperimetric inequality from the Introduction, one can prescribe the length of a closed curve and ask for the shape that minimizes elastic energy. This is known to be a disk for a recent proof see [17] or inspect (1) below in the case  $n = 2$  and notice that it suffices to minimize among convex curves. Alternatively, one can prescribe the enclosed area of a closed curve without intersection points and ask for the shape with minimal elastic energy. That this shape is again a disk was recently shown in [5]. The proof uses the scale invariant functional  $J(\gamma) = \int \kappa^2 ds |\Omega|^{1/2}$ , where  $|\Omega|$  is the area enclosed by  $\gamma$  and a reduction from simply connected sets  $\Omega$  to convex ones. The analogous problems in three dimensions could be: show that among all simply-connected open three-dimensional sets with boundary  $\gamma$ , the ball minimizes the Willmore energy  $\int_{\gamma} H^2 ds$  (with  $H$  denoting mean curvature) for given surface area (or perimeter)  $|\gamma|$  or for given enclosed volume  $|\Omega|$ . However, for  $n = 3$  the Willmore energy alone is scale invariant, so prescribing the perimeter or enclosed volume provides no restriction. This was already noticed in [18], and a simple proof was given in [20] that the minimizing shape must be a sphere. Another generalization to three and more dimensions might be the study of the scale-invariant functional  $J(\gamma) = \int_{\gamma} H^n ds |\Omega|^{1/n}$ . At least among convex sets  $\Omega \subset \mathbb{R}^n$  with boundary  $\gamma$  and for  $n \geq 2$ , one can easily show that

$$\int_{\gamma} H^n ds \geq \int_{\gamma} K^{\frac{n}{n-1}} ds \geq \left( \int_{\gamma} K ds \right)^{\frac{n}{n-1}} |\gamma|^{\frac{-1}{n-1}} = (n\omega_n)^{\frac{n}{n-1}} |\gamma|^{\frac{-1}{n-1}}. \quad (1)$$

Here  $K$  denotes Gauss curvature and  $n\omega_n$  the  $(n-1)$ -dimensional perimeter of the unit sphere in  $\mathbb{R}^n$ . The first inequality uses the geometric-algebraic mean inequality, the second one Hölder's, and for given perimeter  $|\gamma|$  estimate (1) becomes sharp if and only if  $\gamma$  is a sphere. For nonconvex  $\Omega$  and  $n = 3$  a counterexample to (1) was just recently given in [6].

**Open Problem 6: Potential theory of the farthest point distance function**

For  $n = 1$  the function  $u(x) = \frac{1}{2}|x - y|$  is harmonic off  $y$  and  $\Delta u = \delta_y$ .

For  $n = 2$  the function  $u(x) = \frac{1}{2\pi} \log |x - y|$  is harmonic off  $y$  and  $\Delta u = \delta_y$ .

For  $n > 2$  the function  $u(x) = -\frac{1}{n(n-2)\omega_n} |x - y|^{2-n}$  is harmonic off  $y$  and  $\Delta u = \delta_y$ .

In all three cases  $u(x) = \phi(|x - y|)$  is a monotone increasing function of the distance  $|x - y|$ , and  $\Delta u$  is a (nonnegative) probability measure. For  $E \neq \emptyset$  and compact, we define the farthest distance function to  $E$  by

$$d_E(x) := \max_{y \in E} |x - y|.$$

What happens to  $\Delta u$  when  $u(x) = \phi(d_E(x))$  and when  $E$  consists of more than one point  $y$ ? As a maximum of subharmonic functions, the function

$$\phi(d_E(x)) = \phi\left(\max_{y \in E} |x - y|\right) = \max_{y \in E} \phi(|x - y|)$$

is subharmonic; and  $\Delta u(x)$  is still a nonnegative probability measure  $\sigma_E$ , but now it is more regular because  $d_E$  is everywhere positive.

How big is  $\sigma_E(E)$ ? If  $n = 1$ , then  $\sigma_E(E) \leq \sigma_{co(E)}(co(E)) = 1$ , no matter what  $E$  is. Here  $co(E)$  denotes the convex hull of  $E$ . If  $n \geq 2$  and  $E$  is singleton, then  $\sigma_E(E) = 1$ , but if  $E$  is a ball of radius  $R$ , then  $d_E(x) = R + |x|$ , and one can easily calculate that  $\sigma_B(B) = 2^{1-n} < 1$ .

After checking a few more examples, this led Laugesen and Pritsker [12] to conjecture that for any compact  $E \subset \mathbb{R}^n$  with more than one point, we have  $\sigma_E(E) \leq 2^{1-n}$ . The conjecture is still open for  $n \geq 3$ , but for  $n = 2$  it was confirmed in [8]. Moreover, in two dimensions the equality  $\sigma_E(E) = 1/2$  holds whenever  $E$  is a set of constant width. More recently, it was shown in [15] that among sets of constant width the conjecture holds for  $n > 2$ , but in this class the equality  $\sigma_E(E) = 2^{1-n}$  holds only for the ball.

Let me explain why. Formally

$$\sigma_E(E) = \int_E \Delta(\phi(d_E(x))) dx,$$

but  $\sigma_E$  is not absolutely continuous with respect to Lebesgue measure. However, the following integrals are well defined and we were able to show

$$\sigma_E(E) \leq \int_{\partial E} \frac{\partial \phi(d_E(x))}{\partial \nu} ds = \frac{1}{n\omega_n} \int_{\partial E} d_E^{1-n} \frac{\partial d_E}{\partial \nu} ds.$$

For a ball of radius  $R$ , we have  $d_E = 2R$  and the right hand side becomes

$$\frac{1}{n\omega_n(2R)^{n-1}} \int_{\partial E} ds = \frac{1}{n\omega_n(2R)^{n-1}} |\partial E| = \frac{n\omega_n R^{n-1}}{n\omega_n(2R)^{n-1}} = 2^{1-n}.$$

For a set of constant width  $w_E$ , we have  $d_E = w_E$  on  $\partial E$  so that

$$\sigma_E(E) \leq \frac{1}{n\omega_n w_E^{n-1}} |\partial E|,$$

and its perimeter  $|\partial E|$  can be estimated in terms of  $w_E$ . But [9] showed that among all sets of given constant width  $w_E$ , the ball maximizes perimeter. The case of two dimensions is special, because by Barbier's theorem all plane sets of constant width  $d$  have identical perimeter.

There are other partial results, e.g., for polyhedra [21] or point symmetric sets [15] that take more time and space to explain...

**Acknowledgements.** It is a pleasure to thank C.Nitsch for showing me the estimate (1) and G.Thorbergsson for bringing [18,21] to my attention and for providing me with copies of those.

**Open Access.** This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

## References

- [1] H. AUERBACH, Sur un problème de M. Ulam concernant l'équilibre des corps flottants, *Studia Math.* **7** (1938), 121–142.
- [2] W. BLASCHKE, Einige Bemerkungen über Kurven und Flächen von konstanter Breite, *Ber. Verh. Sächs. Akad. Leipzig.* **67** (1915), 290–297.
- [3] G.D. CHAKERIAN AND H. GROEMER, Convex bodies of constant width, in: *Convexity and its Applications*, Gruber, P. M., and Wills, J. M., (eds.), Birkhäuser, Basel, 1983, 4–96.
- [4] L. ESPOSITO, V. FERONE, B. KAWOHL, C. NITSCH, AND C. TROMBETTI, The longest shortest fence and sharp Poincaré-Sobolev inequalities, *Arch. Ration. Mech. Anal.* **206** (2012), 821–851.
- [5] V. FERONE, B. KAWOHL, AND C. NITSCH, The elastica problem under area constraint, *Math. Ann.* **365** (2016), 987–1015.
- [6] V. FERONE, C. NITSCH, AND C. TROMBETTI, On the maximal mean curvature of a smooth surface, to appear in *C.R. Math. Acad. Sci. Paris.* [arXiv:1604.06042](https://arxiv.org/abs/1604.06042).

- [7] N. FUSCO AND A. PRATELLI, On a conjecture by Auerbach, *J. Eur. Math. Soc.* **13** (2011), 1633–1676.
- [8] S.J. GARDINER AND I. NETUKA, Potential theory of the farthest point distance function, *J. Anal. Math.* **101** (2006), 163–177.
- [9] M.A. HERNÁNDEZ CIFRE, G. SALINAS, AND S. SEGURA GOMIS, Two optimization problems for convex bodies in the  $n$ -dimensional space, *Beiträge Algebra Geom.* **45** (2004), 549–555.
- [10] D. HILBERT AND ST. COHN-VOSSEN, *Geometry and the Imagination*, AMS Chelsea, Providence, R.I., 1952, transl. from the German: *Anschauliche Geometrie*, Springer, Berlin, 1932.
- [11] T. LACHAND-ROBERT AND É. OUDET, Minimizing within convex bodies using a convex hull method, *SIAM J. Optim.* **16** (2005), 368–379.
- [12] R.S. LAUGESEN AND I.E. PRITSKER, Potential theory of the farthest-point distance function, *Canad. Math. Bull.* **46** (2003), 373–387.
- [13] B. KAWOHL AND T. LACHAND-ROBERT, Characterization of Cheeger sets for convex subsets of the plane, *Pacific J. Math.* **225** (2006), 103–118.
- [14] B. KAWOHL, Convex sets of constant width, *Oberwolfach Reports* **6** (2009), 390–393.
- [15] B. KAWOHL, C. NITSCH, AND G. SWEERS, More on the potential for the farthest-point distance function, *Potential Anal.* **42** (2015), 699–716.
- [16] B. KAWOHL AND C. WEBER, Meissner’s mysterious bodies, *Math. Intelligencer* **30** (2011), 94–101.
- [17] YU.L. SACHKOV, Closed Euler elasticae, *Proc. Steklov Inst. Math.* **278** (2012), 218–232.
- [18] G. THOMSEN, *Grundlagen der konformen Flächentheorie*, *Abh. Math. Sem. Univ. Hamburg* **3** (1924), 31–56.
- [19] P.L. VÁRKONYI, Neutrally floating objects of density  $\frac{1}{2}$  in three dimensions, *Stud. Appl. Math.* **130** (2013), 295–315.
- [20] T.J. WILLMORE, Note on embedded surfaces, *An. Ști. Univ. “Al. I. Cuza” Iași Sect. I a Mat. (N.S.)* **11B** (1965), 493–496.
- [21] W. WISE, Potential theory and geometry of the farthest distance function, *Potential Anal.* **39** (2013), 341–353.

B. KAWOHL  
Mathematisches Institut,  
Universität zu Köln,  
50923 Köln,  
Germany  
e-mail: kawohl@mi.uni-koeln.de

Received: 8 June 2016